Solving Problems in Irish Mathematics Education

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Introduction

The study reported in this paper addresses the task of enabling preservice teachers in Ireland to experience problem-solving activities in mathematics and problem-based methods of teaching the subject in schools. The issue has been highlighted by the recent revision of the Irish primary school curriculum and current review of the second-level curriculum, and also by the rather undistinguished mathematical performance of Irish 15-year-olds in the OECD PISA 2003 international study – a study which emphasises the solution of problems set in real-life contexts.

The paper describes research on mathematics education for primary teachers in two Irish colleges of education preparing students to teach grades preK-6. It focuses on two courses – one in each college – emphasising mathematical problem solving and problem-based learning. Each author teaches one of the courses. The paper outlines the somewhat contrasting aims, structure and content of the two courses; it then examines selected student outcomes. The authors seek to identify what each can learn from the other with regard to providing courses in keeping with the PISA philosophy.

Theoretical frameworks

Theoretical frameworks for this paper are drawn from three sources: the PISA mathematics framework; the structure of typical problem-solving courses in mathematics teacher education; and Shulman's analysis of teacher knowledge.

The *PISA mathematics framework* is concerned with assessing how well students can use the mathematics they have learned in realistic situations (mathematical literacy) (OECD, 2003). It reflects current trends towards emphasising the processes of problem solving and mathematisation, and consequently towards problem-based teaching and learning. The framework has three dimensions: context, content, and competency.

- Problem contexts are categorised as Personal, Social/Occupational, Public, and Scientific. In the first cycle of assessment for PISA ("PISA 2000"), a further category was used: Intra-mathematical, explicitly accommodating situations in which the context for the question was provided by mathematics itself.
- The mathematical content is described in terms of four "overarching ideas": Quantity; Space and Shape; Change and Relationships; and Uncertainty.
- The competency dimension is divided into three competency classes. These are: Reproduction, which typically involves performing routine calculations; Connections, which deals for example with integrating information or solving problems using familiar procedures in contexts; and Reflection, which requires skills such as non-routine problem solving and making mathematical arguments and generalisations.

While PISA may have given fresh impetus to the approach, *problem solving in mathematics teacher education* has been a focus of interest for many years. Courses described in the literature are typically designed on the assumption that teachers are unlikely to implement problem-solving or similar approaches effectively unless they have experienced these as learners and have also reflected on the experience (see for example Even and Lappan 1994, Lester et al. 1994, Cooney 2001, Ponte 2001). Lester et al. (1994), in particular, note the importance of teachers engaging in reflective writing.

Also with regard to teacher education, Shulman's (1986, 1987) *analysis of teacher knowledge* is useful in distinguishing between content knowledge (CK) and pedagogical content knowledge (PCK). PCK involves "that special amalgam of content and pedagogy that is uniquely the province of teachers" (Shulman, 1987, 8), and is an obvious focus of "methodology" courses. The extent to which CK belongs in such courses, or should be addressed in "mathematics" courses, or both, is a matter for debate.

Context of the study

A revised school curriculum for Irish primary schools was published in 1999 (DES/NCCA 1999). The mathematics component, which was implemented in 2002, is intended to reflect the PISA philosophy in that it emphasises the solution of problems set in real-life contexts. However the second-level mathematics curriculum currently gives more emphasis to abstract mathematics, and many of the problems addressed are intra-mathematical (NCCA 2005). Thus, if Irish students entering teacher education programmes are going to implement the primary mathematics curriculum successfully, they are likely to have to change their view of mathematics from that developed in the course of their second-level schooling.

As indicated above, their teacher education courses have a crucial role to play in this respect; courses involving mathematical problem solving must be a priority. This paper aims to compare and contrast the aims, structure, content and outcomes of two such courses: one taught by each author in his or her own college. With regard to student outcomes, the paper focuses in particular on the students' written reflections.

Description and comparison of the courses

This section describes the two courses. Each is part of an eighteen-month programme undertaken by graduates in order to qualify as primary teachers. The course taught by the second author at St. Patrick's College explicitly uses a problem-based approach, and is outlined in section 4.1; that taught by the first author at Froebel College – with a somewhat wider brief – is outlined in section 4.2.

The St. Patrick's College course

Aims and structure of the course

At present, pre-service student teachers in the St. Patrick's College Post-Graduate Diploma in Primary Teaching programme receive a two-module course in mathematics education in which the main aim is to develop student-teachers' confidence and competency in teaching the primary mathematics curriculum using a problem-based approach. Students are expected to familiarise themselves with the structure of the primary mathematics curriculum and be able to select or design activities/tasks for any particular content strand or process skill category. The modules concentrate on developing PCK. The students are not in receipt of any formal coursework to improve their CK in mathematics.

The second module of the course concerns the teaching of the mathematics curriculum in Third through Sixth Classes (for children typically aged 8 to 12). The sessions are all mainly of workshop format with an appropriate balance of activity, discussion, questioning and explanation. There are approximately thirty students per workshop group.

Content and methodology of the course

The content and methodology of four (of ten) two-hour workshops in Module 2 (Week 5, Week 7, Week 9, and Week 10), which were the focus of this study, are set out in Table 1 below. These four workshops address the four PISA overarching ideas (Quantity, Space and Shape, Uncertainty, and Change and Relationships), though the correspondence is not exact; the structure reflects that of the Irish primary curriculum. The problems addressed by the students were designed to mirror the kinds of problems used in PISA; in general they fall into the Connections competency class. Two of them are presented in the Appendix to this paper.

Course assessment

The assignment for this course consisted of writing up six workshop journals. In these journals students were asked to reflect on the workshops in terms of:

- how their own mathematical thinking has been challenged
- what they have learned about the process of learning and teaching mathematics.

The latter obviously refers to PCK; the former includes elements of CK. The responses to the first task (what they have learned about their own mathematics) in journals submitted by each of the students in the course formed the data used to comment on the St. Patrick's College course.

Table 1: St. Patrick's College – Content and methodology of the Mathematics Education Module 2 workshops

Session	Content	Methodology
Week 5. Fractions, decimals, and percent – Part II	» Solve problems involving operations with decimals and percentages » Compare and order decimal fractions and percentages » Express decimal fractions in both decimal and percentage form	Decimal and percent problems in realistic contexts (e.g. pizzas for lunch) Show decimals and % s on a 10x10 grid Relate fractions to decimals and percent on squared paper and number lines Calculator activities to explore fractions, decimals and percent
Week 7. Measure	» Internalisation of standard units and estimation and measurement procedures: » Length: meter, centimetre, kilometre, scale » Area: cm2 and m2, hectare » Weight and Capacity » Volume of cuboid » Time: digital and analogue 24hr clock time; time zones, speed » Money: currencies	Use of problems to investigate aspects of measurement, (e.g., designing room to scale, planning holiday, packaging, estimating area from maps) Exploration of relationship between perimeter and area Deduction of formulae, e.g., circumference, volume of cuboid Activities to investigate time, speed, distance
Week 9. Probability and Chance	» Use vocabulary of certainty and chance » Order events in terms of likelihood of occurrence » Identify and list all possible outcomes of simple random processes » Estimate the likelihood of occurrence of events » Construct and use frequency charts and tables	Explore concepts of probability in a problem contexts (e.g. ages of children in a family) List everyday events in order of likelihood of occurrence Fairness and bias – make own spinners for games Probability experiments, e.g., coin tossing, dice throwing. Compare theoretical predictions with experimental results
Week 10. Number sentences and introduction to algebraic concepts	» Number patterns (predict and generalise) » Translate problems to symbolic form (with variable) and vice versa (construct and interpret mathematical models) » Functions. Variables. Equations » Directed numbers	Mathematisation of problems situations using algebraic notation (e.g. The burger stand) Geometric number patterns. Generalisation of patterns in context Make up problems for number sentences. Make up number sentences for number problems Functions – "guess my rule" Misconceptions about variables Using formulae and brackets Notion of balance in equations Negative numbers in context, e.g., temperature scale

The Froebel College course

Aims and structure of the course

The programme for the Higher Diploma in Education (Primary) at Froebel College includes a compulsory one-semester module in mathematics as well as a two-semester course in mathematics teaching methods. The mathematics module is discussed in this paper.

When the course began in 1995, it was intended to provide students with an experience of learning the subject *at their own level*, and hence to contribute to their own academic and personal development as well as addressing their needs as future primary teachers. Because of the very varied mathematical backgrounds of the students and the problem-solving emphasis in the (then forthcoming) revised primary curriculum, the course aimed to address the nature of mathematics and students' attitude to the subject as well as specific mathematical content. Hence, two major (and intertwined) threads in the course were *problem-solving approaches* ("*mathematical thinking*") and *activities likely to promote reflection* (on the students' own experiences of mathematics and on mathematics itself). The mathematical thinking element was addressed largely through "investigations" – open-ended problems typically capable of being developed in more than one way, yielding more than one acceptable solution, and approachable at many levels (to allow students to work at their own level of comfort, whether with primary school mathematics or with more advanced content and strategies). The reflection element of the course was addressed by two types of student activity: the writing of autobiographical essays outlining their attitudes to mathematics at different stages in their school careers, and the completion of comment sheets and questionnaires asking about their experiences during the course.

The *content of the primary curriculum* was initially only a minor thread in the course. Students revised CK and skills by using them as needed in the investigations. Latterly, however, concerns about students' CK with regard to primary mathematics and their confidence in using it have led to a change in emphasis. Primary curricular topics considered most likely to be problematic are now covered explicitly.

Content and methodology of the course

The content and methodology of the course as planned for 2004-05 are presented in Table 2. As lecture sessions were of different lengths and styles at different times of the year, the table maps out the main blocks and incorporates the varying forms of class organisation.

Table 2: Froebel College – Content and methodology of the course

Section	Content	Methodology
1	Introduction	Exposition / discussion
2	Mathematics and myself	Reflective writing: autobiographical essay
3	Content of the curriculum: - algebra: algebra of number: priority of operations, calculator use; number sentences (notion of balance in equations), variables, functions, patterns - probability and chance: ordering events in terms of likelihood of occurrence, relative frequency and estimated probability, outcomes of a random process, theoretical probability for equally likely outcomes - measure: length; perimeter, area, volume; relationships between perimeter and area and between area and volume (e.g. maximising area for fixed perimeter)	Whole-class teaching; problem solving / exposition / discussion / games (e.g. "Guess my rule") / practice
4	Mathematical thinking via problems involving: - perimeter, area and volume - number - data pattern generalisation	Small-group work doing investigations
5 (after the end of the course)	Mathematics and myself	Reflective writing: completion of reflective questionnaire

The "curriculum content" section, as with the course at St. Patrick's College, reflects the structure and content of the primary curriculum. For the investigations ("mathematical thinking"), the categorisation in terms of "perimeter, area and volume," "number" and "data pattern generalisation" may appear to correspond with the PISA overarching ideas Space and Shape, Quantity, and Change and Relationships respectively, but in practice once again the correspondence is not exact; the number investigations, in particular, also include elements of Change and Relationships (even leading to the development of proofs). One such problem, Special Sums, is shown in the Appendix. As regards competency class, all the investigations require some form of behaviour of at least Connections type, though some can be addressed initially in a fairly routine way that calls on Reproduction skills. It should be noted that, reflecting the focus of the course on the nature of mathematics, about half of the investigations offered to the students are intra-mathematical.

While the main focus is on CK and attitudes to (or beliefs about) mathematics, discussions about teaching methodology arise naturally and are encouraged. Moreover, it is hoped that the students learn experientially how they might teach. Thus, the course addresses PCK as well as CK.

Course assessment

The course is assessed by means of a portfolio, which in 2004-05 contained as its main items:

- an autobiographical essay on the students' experiences of and attitudes to mathematics over the years
- written accounts ("write-ups") of investigations of the students' choice, describing the process as well as displaying the product of their work
- a final reflective questionnaire.

The record of the investigations selected by the students for their portfolios, and their final questionnaires, provide the data used to comment on the Froebel College course.

Implementation of the courses and students' reflections

St. Patrick's College

Thirty graduate students (almost all in their twenties and nine male) attended the Module 2 workshops in the Second Semester 2005 (January to May). The students normally worked in twos or threes on problems and activities.

The journals submitted by the students in relation to the four workshops mentioned in Table 1 above (about eighty journals in all) were analysed in terms of their responses to the question "How has your mathematical thinking been challenged?" For each student the principal themes of his or her responses with regard to the workshops were listed and classified. The results are shown in Table 3.

Froebel College

Twenty-nine graduate students (twenty-five women and four men) completed the Higher Diploma course in 2004-2005; they attended the mathematics course in the second semester (Autumn 2004 to May 2005, with two breaks for teaching practice). Most students were in their twenties, but a few were considerably older.

Ten of the eighteen hours were devoted to primary curricular topics, with problem-based or investigative methods being used intermittently through this period. Typically the topics were introduced by means of challenges or problems but reinforced by routine reproduction-type exercises. The final five hours were explicitly focused on investigations. Students selected investigations that interested them and worked in groups of about three. The lecturer's role during this period was

Table 3: St. Patrick's College – Classification of student responses to the question "How has your own mathematical thinking been challenged?"

Main aspect of response	Specific aspect of response	Number of students
1. Making mathematical connections	Fractions/Decimals/Percent Arithmetic/Algebra Problem sections Modes of representation	11
2. Multiple solution methods	Different methods of solution Looking for best method of solution Personal methods of solution	10
3. Experimental approach to mathematics	Experimental aspect of measure/data Mathematical prediction vs experimental prediction	8
4. Role of formulae	Dependency on formulae/mnemonics Priority of operations Appropriateness of formulae	7
5. Mathematical reasoning and understanding	Misconceptions (fractions, perimeter and area relationships) Lack of understanding Algebraic thinking vs algebraic methods	7
6. Communicating mathematics	Problem solving processes Justifying methods	5
7. Real world problems and contexts	Everyday contexts Integrating mathematics with geography	4
8. Estimation process		4
9. Chance/probability		3
10. Working collaboratively on mathematics		2
11. Reading and interpreting a problem		2
12. Fear of problem-solving		1

not only to encourage the development of problem-solving strategies but also to find "teachable moments" with regard to the CK that the investigations were intended to revise.

Most students submitted two investigation write-ups to their portfolios; some submitted more, and one group availed of the option to present just one investigation studied in depth. In total, sixty-three write-ups were received. The categories into which the investigations fell are shown in Table 4 below. The reflective questionnaires completed at the end of the course contained very rich accounts of the students' thinking about their own learning, mathematics education in general, and their experiences of the course in particular. Many of the issues arising in the St. Patrick's students' reflective journals were mentioned. However, several of these were raised explicitly by questions that asked students to discuss (for example) advantages and disadvantages of group work, or personal positive and negative reactions to open-ended problems and a lack of unique right answers; so occurrences were not spontaneous, and data from such questions do not lend themselves to frequency counts as done above. The analysis therefore focuses on two questions: "Which investigation(s) did you find easiest to start, and why?" and "Which investigation(s) did you eventually find most satisfying, and why?" Responses to these questions were received from twenty-seven students; they are summarised in Table 4.

Table 4: Froebel College – Classification of investigations selected for portfolios, and of responses to questions on investigations found *easiest to start* and found *most satisfying*

Type/problem	Occurrences in portfolios	Easiest to start		Most satisfying	
		Number of students	Main reasons	Number of students	Main reasons
Number - Special Sums - Other	 22 15	4 10 4	Comfort with numbers; clear starting points, step-by-step approach	1 8 5	Found pattern; persisted; solved problem, identified reasons, supplied proof
Perimeter/ Area/ Volume	9	4	Involved drawing; real life	6	Real life
Data Pattern Generalisation	17	2	Materials to handle; visual	8	Challenge, eventual success; fascination with problem or pattern
Other*	-	3	Various	1	Practical
TOTAL	63	27		29**	

**** Some students listed characteristics without indicating the investigation or group of investigations in which these occurred***

***** One student named three investigations jointly (Special Sums and two Area/Volume investigations)***

Discussion

First, it is clear that despite the different roles of the two courses, they shared many aspects. Both courses contain the elements of problem-solving experience and reflection that are emphasised in the literature with regard to promoting problem-solving approaches in schools.

The problems addressed at St. Patrick's College are more PISA-like in style; they can be tackled in a short period, and although addressable by different approaches they generally have unique right answers. The focus on solving problems set in contexts is the central feature of the course. The investigations used in the Froebel College course reflect a somewhat different tradition. Some are open-ended, allowing students to explore and develop the work in a way that interests them; some can lead to general results for which the more able students can provide formal proofs; typically they may occupy one or two full periods of work. The greater emphasis on intra-mathematical problems has already been noted. Perhaps the approach at St. Patrick's would transfer more easily to the ordinary classroom (although some Froebel students reported using investigations on teaching practice); perhaps the more philosophical Froebel approach raises deeper questions about the nature of mathematics itself.

With regard to reflection, the assignments for the two courses asked somewhat different questions of the students and therefore got somewhat different answers, but many common themes can be identified. In both cases, the reflective element contributed greatly to the students' final grade for the course. It is perhaps worth noting that students could achieve good marks through their ability to reflect on their learning without necessarily displaying great mathematical competence.

Similarities between the courses extend to the CK explicitly or tacitly addressed. While this is not surprising (given that the Froebel College course now has a specific focus on CK for the primary curriculum), some of the activities and problems used in the two courses are notably alike. However, perhaps many other courses, whether dealing with mathematics or methodology, also use such approaches. The relationship between CK and PCK in the courses is somewhat contrasting, reflecting their different roles. The Froebel College course starts with greater explicit focus on CK, but probably ends with less focus as general aspects of the nature of mathematics are considered; the students' final reflections often referred to PCK issues. The St. Patrick's students, although chiefly addressing PCK, actually reflect on CK issues (in the context of "mathematical thinking") as a major part of their assignment. At a time at which there are concerns about Irish student-teachers' CK, questions arise – and are not resolved in this paper – as to whether CK is best developed in the context of a course that focuses mainly on PCK, or whether CK can be most satisfactorily addressed when students are free to concentrate on their own learning and PCK is a subsidiary issue.

Conclusion

The revision and implementation of mathematics curricula in keeping with the PISA philosophy is dependent to a great degree on changing teachers' knowledge, beliefs and attitudes to mathematics and mathematics learning and pedagogy. Irish student teachers have been exposed in their schooling to a more traditional curriculum; they therefore tend to teach mathematics in the way they themselves were taught, and they find it difficult to conceptualise, much less implement, a more PISA-like philosophy of mathematics and mathematics education. The issue is not unique to Ireland. If a PISA-like approach is to be adopted, it is the responsibility of the teacher education programmes in universities and colleges of education to provide learning environments in which student teachers can experience mathematics as problem solving, reflect on the experience, and experiment with problem-based methods of teaching mathematics during their teaching practice. This paper illustrates ways in which both "mathematics" and "methodology" courses can provide students with appropriate experiences and opportunities for reflection.

The authors learnt much by comparing and contrasting their courses. It was satisfying to find that similar issues were being addressed, albeit in somewhat different ways that reflected the different roles of the courses. Each author now intends to make use of some of the problems presented in the other author's course in order to widen their students' experience of problem-solving approaches. Hopefully the ultimate beneficiaries will be children in primary schools.

Appendix

Specimen problems used in the courses, classified according to the PISA framework by their main context, content area and competency class

St. Patrick's College
Problem 1 (Public; Space and Shape; Connections):
Estimate the area of Australia in square km using a map and scale (1cm = 200 km). Resources available: squared paper, string and scissors, cm cubes, 2cm cubes.

Problem 2 (Personal; Quantity; Connections):
Mum, Dad, Peter and Ann had pizzas for lunch. The first pizza was shared equally among them. Mother then cut the second pizza into four parts also, but said, "I've had enough. You three can share this one." But Ann said "No, one of these pieces is enough for me." Mum then turned to Father and Peter and said "You two can share the other three pieces." How much pizza did each person eat?

Froebel College
Special sums (Intra-mathematical; Change and Relationships; Connections/Reflection):
Pick three different digits (three from 0, 1, 2, … , 9) and find their sum.
Using your three digits, form as many two-digit numbers as possible. Do not use a digit twice in your numbers. How many can you get? (Notes: if you picked 0, then count 03 (say) as a two-digit number.)
Now find the sum of the two-digit numbers, and divide it by the sum of the original three digits. What happens? Does it *always* happen? Can you justify this result or even prove it?
Try for four digits. What will happen for five digits?
[Students are encouraged to continue and to justify and prove as general a result as possible.]

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